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Exact Solution of the Derbenev-Kondratenko $\hat{n}$ Axis for a Model with One Resonance

S. R. Mane
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

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Fermilab, P.O. Box 500, Batavia, Illinois 60510

Let us call the quantization axis of the spin eigenstates of the Hamiltonian of a particle circulating in a storage ring $\hat{n}$, i.e. the operator $\vec{s}.\hat{n}$ commutes with the Hamiltonian: $\{\vec{s}.\hat{n}\,,\mathcal{H}\} = 0$, where $\{\,,\,\}$ denotes a Poisson Bracket. The defining properties of $\hat{n}$ were given by Derbenev and Kondratenko,[1] and are (i) $\hat{n}$ satisfies the equation of spin motion

$$\frac{d\hat{n}}{d\theta} = \vec{\Omega} \times \hat{n}\ , \tag{1}$$

where $\vec{\Omega}$ is the spin precession vector, and (ii) $\hat{n}$ satisfies the periodicity conditions

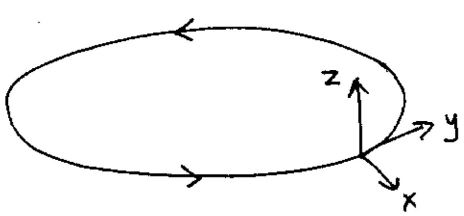
$$\hat{n}(I,\psi,\theta) = \hat{n}(I,\psi+2\pi,\theta) = \hat{n}(I,\psi,\theta+2\pi)\ . \tag{2}$$

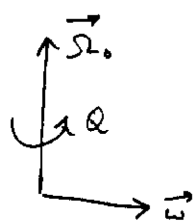
Here $\{I,\psi\}$ denotes the set of orbital action-angle variables, i.e. $\hat{n}$ depends on the orbital trajectory. Thus there is not just one value of $\hat{n}$ for all trajectories, but an infinite set, one for each value of $\{I,\psi\}$.

The above definition is rather abstract, and in general $\hat{n}$ has been calculated exactly only on the closed orbit of a storage ring. Perturbation theory has had to be used to find $\hat{n}$ on other trajectories.[2] There is a simple non-trivial model, however, where one can solve for $\hat{n}$ exactly, and that is the point of this note.

The model is a vertical static field plus a horizontal field rotating at tune Q. We decompose $\vec{\Omega}$ in Eq. (1) into $\vec{\Omega} = \vec{\Omega}_0 + \vec{\omega}$, where $\vec{\Omega}_0$ is the value on the closed orbit and $\vec{\omega}$ is the additional term due to an orbital oscillation. Then

$$\begin{aligned}\vec{\Omega}_0 &= \nu\hat{z} \\ \vec{\omega} &= \epsilon\,[\,\hat{x}\cos\psi + \hat{y}\sin\psi\,] = \epsilon\,[\,\hat{x}\cos(Q\theta+\psi_0) + \hat{y}\sin(Q\theta+\psi_0)\,]\ .\end{aligned} \tag{3}$$





The vertical field is the main field that makes the particles circulate around the ring, and $\vec{\omega}$ is the perturbation due to spin-orbit coupling. Here ψ_0 is the initial phase of the orbital oscillation, and ϵ describes the strength of the spin-orbit coupling. The coordinate system is $\hat{x}$ radial, $\hat{y}$ longitudinal and $\hat{z}$ vertical. A positive rotation is counterclockwise.

The above model cannot be constructed exactly in a storage ring, because real magnets produce more than one harmonic in $\vec{\omega}$. However, the above model is still a valid Hamiltonian system, and is a good approximation near the resonance $\nu = Q$.

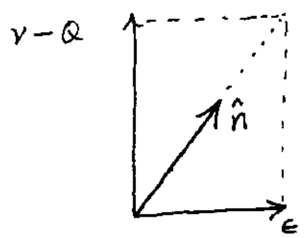
Let us first review the case where $\vec{\Omega} = \vec{\Omega}_0$ only. Then the spin precesses around $\vec{\Omega}_0$, which is a constant vector, and the solution for $\hat{n}$ is obvious: it is just $\hat{n} \parallel \vec{\Omega}_0$, i.e. $\hat{n} = \hat{z}$.

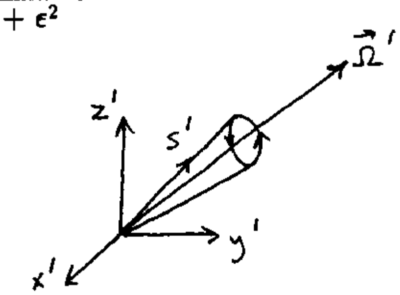
We now consider $\vec{\Omega} = \vec{\Omega}_0 + \vec{\omega}$. We first transform to a frame rotating counterclockwise at tune Q around $\hat{z}$. Then the spin precession tune around $\hat{z}$ becomes $\nu - Q$ and $\vec{\omega}$ becomes stationary. Using primes to denote vectors in the new frame,

$$\vec{\Omega}' = (\nu - Q)\hat{z} + \epsilon\,[\hat{x}'\cos(\psi_0) + \hat{y}'\sin(\psi_0)] \equiv (\nu - Q)\hat{z} + \vec{\epsilon}\,' \ . \tag{4}$$

In this frame $\vec{s}\,'$ precesses around $\vec{\Omega}'$, which is a constant. The solution for $\hat{n}$ is $\hat{n} \parallel \vec{\Omega}'$, i.e.

$$\hat{n} = \frac{(\nu - Q)\hat{z} + \vec{\epsilon}\,'}{\sqrt{(\nu - Q)^2 + \epsilon^2}} \ . \tag{5}$$





We therefore see that $\hat{n}$ is the spin rotation axis *in the frame where the Hamiltonian is stationary*. The diagonalized Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{orb} + \vec{s}\,'.\vec{\Omega}' \\ &= QI + \sqrt{(\nu - Q)^2 + \epsilon^2}\,\vec{s}\,'.\hat{n} \end{aligned} \tag{6}$$

In other frames, where $\mathcal{H}$ is not stationary, $\vec{s}.\hat{n}$ still commutes with the Hamiltonian in that frame, because of the invariance of Poisson Brackets under canonical transformations, i.e. $\{\vec{s}.\hat{n}\,,\mathcal{H}\} = 0$ in all frames related to the above by a canonical transformation (a proof of this statement is given in Ref. 3). (Here $\mathcal{H}$ means the Hamiltonian in the frame after the canonical transformation.) In the original reference frame, the solution is

$$\hat{n} = \frac{(\nu - Q)\hat{z} + \epsilon\,[\,\hat{x}\cos\psi + \hat{y}\sin\psi\,]}{\sqrt{(\nu - Q)^2 + \epsilon^2}} \ . \tag{7}$$

This obviously satisfies the periodicity conditions Eq. (2). Notice that $\hat{n}$ is not parallel to $\vec{\Omega}$. The vector $\hat{n}$ is constrained to be a unit vector in all frames, whereas $\vec{\Omega}$ is not. Hence they transform differently under canonical transformations. The Derbenev-Kondratenko definition (Eqs. (1) and (2)) allows one to calculate $\hat{n}$ purely in terms of functions specified in one frame, without the need for canonical transformations, which can be complicated, in general.

The above model also helps one to understand the distribution of spin eigenstates in equilibrium, e.g. in a high-energy electron storage ring. When $\vec{\Omega} \parallel \hat{z}$ only, all the eigenstates are quantized vertically, and so the polarization will be vertical also. When $\vec{\Omega}$ is *not* the same for all trajectories, the spin states will be quantized along $\hat{n}$ for each trajectory, because that is the spin precession axis when the Hamiltonian is stationary. The equilibrium distribution of spins will be a cone of vectors $\langle\hat{n}\rangle$, where the average is over the equilibrium distribution of orbital actions and angles. The "local polarization" for particles in a small phase-space volume element $dI\,d\psi$ around $\{I,\psi\}$ will thus point along $\hat{n}$, and the "global polarization" is given by the phase-space average

$$\vec{P}_{eq} = \int f(I,\psi)\,\langle\vec{s}.\hat{n}\rangle\,\hat{n}\;dI\,d\psi\;, \tag{8}$$

where f is the probability density function of particles in orbital phase space and $\langle\vec{s}.\hat{n}\rangle$ is the average spin projection of particles along $\hat{n}$ in the volume element $dI\,d\psi$ around $\{I,\psi\}$.

ACKNOWLEDGEMENTS

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1 Ya.S. Derbenev and A.M. Kondratenko, Zh. Eksp. Teor. Fiz. **62**, 430 (1972) [Sov. Phys. JETP **35**, 230 (1972)].

2 Perturbative algorithms to calculate $\hat{n}$ can be found in K. Yokoya, Particle Accelerators **13**, 85 (1983), S.R. Mane, Phys. Rev. A **36**, 120 (1987) and K. Yokoya, Nucl. Instrum. Meth. **A**258, 149 (1987).

3 H. Goldstein, "Classical Mechanics," Addison-Wesley (1981).